International Journal of Theoretical Physics, Vol. 37, No. 12, 1998

Energy Spectrum of a Two-Parameter Deformed Hydrogen Atom

Zhao-xian Yu[1] and Ye-hou Liu[2]

Received March 13, 1998

Utilizing the dynamic symmetry of the two-parameter deformed (q, s-deformed) quantum group $SO(4)_{q,s}$, the q, s-deformed hydrogen atom is transformed into a 4-dimensional q, s-deformed isotropic oscillator subjected to a constraint condition, and the energy spectrum of the q, s-deformed hydrogen atom is derived.

The Hamiltonian of a 3-dimensional hydrogen atom in the center-of-mass frame is given by

$$H = \frac{-\hbar^2}{2\mu} \nabla^2 - \frac{e^2}{r} \tag{1}$$

which means that the hydrogen atom has the dynamic symmetry of $SO(4)$ group. Introducing the Runge–Lenz vector operators

$$\vec{A} = \frac{\vec{r}}{r} + \frac{1}{2\mu e^2} (\vec{L} \times \vec{p} - \vec{p} \times \vec{L}) \tag{2}$$

we have

$$[H, \vec{L}] = [H, \vec{A}] = 0 \tag{3}$$

$$\vec{L} \cdot \vec{A} = \vec{A} \cdot \vec{L} \tag{4}$$

[1] Shengli Oilfield TV University, Shandong Province 257004, China.
[2] Chongqing Petroleum Advanced Polytechnic College, Chongqing 400042, China.

0020-7748/98/1200-297 9$15.00/0

It is easy to find that $SO(4)$ group can be constructed via $\bar{L}$ and $\bar{A}$. We define two new operators

$$\bar{J} = (\bar{L} + \bar{A})/2, \qquad \bar{K} = (\bar{L} - \bar{A})/2 \tag{5}$$

The $SO(4)$ group is equivalent to $SO(3, \bar{J}) \otimes SO(3, \bar{K})$, and the following commutative relations hold:

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_0 \tag{6}$$

$$[K_0, K_\pm] = \pm K_\pm, \qquad [K_+, K_-] = 2K_0 \tag{7}$$

The Jordan–Schwinger realizations of $\bar{J}$ and $\bar{K}$ can be obtained from the four independent bosonic oscillators:

$$J_+ = a_1^+ a_2, \qquad J_- = a_2^+ a_1, \qquad J_0 = (N_1 - N_2)/2 \tag{8}$$

$$K_+ = a_3^+ a_4, \qquad K_- = a_4^+ a_3, \qquad K_0 = (N_3 - N_4)/2 \tag{9}$$

where $a_i^+ a_i = N_i$, $a_i a_i^+ = N_i + 1$ (for $i = 1, 2, 3, 4$), and

$$[a_i, a_i^+] = 1, \qquad [N_i, a_i] = -a_i, \qquad [N_i, a_i^+] = a_i^+ \tag{10}$$

It is well known that the 3-dimensional hydrogen atom is equivalent to a 4-dimensional oscillator subjected to a constraint condition (Kustaanheimo and Steifel, 1965; Gerry, 1986; Kibler and Négadi, 1991). The constraint condition is equivalent to Eq. (4); one has

$$\bar{J}^2 = \bar{K}^2 \tag{11}$$

The Hamiltonian of the 4-dimensional oscillator is

$$\mathcal{H} = \frac{1}{2}\hbar\omega \sum_{j=1}^{4} (a_j^+ a_j + a_j a_j^+) \tag{12}$$

where $\omega = \sqrt{-E/2\mu}$, and E is the energy of hydrogen atom. The eigenvalue of $\mathcal{H}$ is given by

$$\mathcal{E} = \hbar\omega\left(\sum_{i=1}^{4} n_i + 2\right) = e^2 \tag{13}$$

From Eq. (11), one have the constraint condition

$$n_1 + n_2 = n_3 + n_4 \tag{14}$$

with n_i ($i = 1, 2, 3, 4$) the eigenvalue of operator $\overset{+}{a_i} a_i$. Therefore the energy spectrum of a 3-dimensional hydrogen atom is

$$E = \frac{-\mu e^4}{2n^2\hbar^2} \tag{15}$$

with $n = n_1 + n_2 + 1 = n_3 + n_4 + 1$. the eigenstate of the hydrogen atom in the occupation number representation is

$$|n\rangle = |n_1\rangle|n_2\rangle|n_3\rangle|n_4\rangle = |n_1\rangle|n_2\rangle|n_3\rangle|n_1 + n_2 - n_3\rangle \tag{16}$$

where

$$|n_i\rangle = ((\overset{+}{a_i})^{n_i}/\sqrt{n_i!})|0\rangle \tag{17}$$

$$|n_1 + n_2 - n_3\rangle = \frac{(\overset{+}{a_1} + \overset{+}{a_2} - \overset{+}{a_3})^{n_1+n_2-n_3}}{\sqrt{(n_1 + n_2 - n_3)!}}|0\rangle \tag{18}$$

We now construct a q, s-deformed hydrogem atom. We generalize Eq. (4) to the q, s-deformed case, i.e.,

$$\bar{L}_{qs} \cdot \bar{A}_{qs} = \bar{A}_{qs} \cdot \bar{L}_{qs} \tag{19}$$

Correspondingly, a close q, s-deformed quantum group $SO(4)_{q,s} \sim SU(2)_{qs} \otimes SU(2)_{qs}$ can be formed via vectors $\bar{L}_{qs}$ and $\bar{A}_{qs}$. We define

$$\bar{J}' = (\bar{L}_{qs} + \bar{A}_{qs})/2, \qquad \bar{K}' = (\bar{L}_{qs} - \bar{A}_{qs})/2 \tag{20}$$

It is easy to check that $\bar{J}'$ and $\bar{A}'$ satisfy the commutative relations of the q, s-deformed quantum group $SU(2)_{qs}$ (Jing and Cuypers, 1993) and

$$[J'_0, J'_\pm] = \pm J'_\pm, \qquad s^{-1}J'_+J'_- - sJ'_-J'_+ = s^{-2J'_0}[2J'_0] \tag{21}$$

$$[K'_0, K'_\pm] = \pm K'_\pm, \qquad s^{-1}K'_+K'_- - sK'_-K'_+ = s^{-2K'_0}[2K'_0] \tag{22}$$

and we get

$$\bar{J}'^2 = s^{2J'_0}(s^2J'_-J'_+ + [J'_0]_{qs}[J'_0 + 1]_{qs}) \tag{23}$$

$$\bar{K}'^2 = s^{2K'_0}(s^2K'_-K'_+ + [K'_0]_{qs}[K'_0 + 1]_{qs}) \tag{24}$$

where we have used the notation $[x]_{qs} = s^{1-x}[x] = s^{1-x}(q^x - q^{-x})/(q - q^{-1})$.

In order to obtain the Jordan–Schwinger realization of the q, s-deformed vectors $\bar{J}'$ and $\bar{K}'$, we introduce four independent q, s-deformed bosonic oscillators $\{a'_i, \overset{+}{a'_i}, N'_i\}$ ($i = 1, 2, 3, 4$) (Jing and Cuypers, 1993):

$$a_1'^{+}a'_1 = [N'_1]_{qs}, \qquad a'_1a_1'^{+} = [N'_1 + 1]_{qs} \tag{25}$$

$$a_2'^{+}a'_2 = [N'_2]_{qs^{-1}}, \qquad a'_2a_2'^{+} = [N'_2 + 1]_{qs^{-1}} \tag{26}$$

$$a_3'^{+}a_3' = [N_3']_{qs}, \qquad a_3'a_3'^{+} = [N_3' + 1]_{qs} \tag{27}$$

$$a_4'^{+}a_4' = [N_4']_{qs^{-1}}, \qquad a_4'a_4'^{+} = [N_4' + 1]_{qs^{-1}} \tag{28}$$

The following relations hold:

$$a_1'a_1'^{+} - s^{-1}qa_1'^{+}a_1' = (sq)^{-N_1'}, \qquad a_3'a_3'^{+} - s^{-1}qa_3'^{+}a_3' = (sq)^{-N_3'} \tag{29}$$

$$a_2'a_2'^{+} - sq^{-1}a_2'^{+}a_2' = (sq)^{N_2'}, \qquad a_4'a_4'^{+} - sq^{-1}a_4'^{+}a_4' = (sq)^{N_4'} \tag{30}$$

with the notation $[x]_{qs^{-1}} = s^{x-1}[x]$.

The Jordan–Schwinger realizations of vectors $\bar{J}'$ and $\bar{K}'$ can be written as

$$J_+' = a_1'^{+}a_2', \qquad J_-' = a_2'^{+}a_1', \qquad J_0' = (N_1' - N_2')/2 \tag{31}$$

$$K_+' = a_3'^{+}a_4', \qquad K_-' = a_4'^{+}a_3', \qquad K_0' = (N_3' - N_4')/2 \tag{32}$$

It is easy to prove that Eqs. (31)–(32) satisfy Eqs. (21)–(22).

From the above results, the Hamiltonian of a 4-dimensional q, s-deformed oscillator is

$$\mathscr{H}' = \frac{1}{2}\hbar\omega_{qs}\sum_{j=1}^{4}(a_j'^{+}a_j' + a_j'a_j'^{+}) \tag{33}$$

$$\omega_{qs} = \sqrt{-E_{qs}/2\mu} \tag{34}$$

where E_{qs} stands for the energy of the q, s-deformed hydrogen atom.

We can define the vacuum state $|0\rangle$ from $a_i'|0\rangle = 0$:

$$a_i'^{+}|n_i\rangle_{qs} = \sqrt{[n_i + 1]_{qs}}|n_i + 1\rangle_{qs}, \qquad a_i'|n_i\rangle_{qs} = \sqrt{[n_i]_{qs}}|n_i - 1\rangle_{qs} \qquad (i = 1, 3) \tag{35}$$

$$a_i'^{+}|n_i\rangle_{qs} = \sqrt{[n_i + 1]_{qs}}|n_i + 1\rangle_{qs}, \qquad a_i'|n_i\rangle_{qs} = \sqrt{[n_i]_{qs^{-1}}}|n_i - 1\rangle_{qs} \qquad (i = 2, 4) \tag{36}$$

So we have the eigenvalue of Eq. (33),

$$\begin{aligned}\mathscr{E} = \frac{1}{2}\hbar\omega_{qs}\{&[n_1 + 1]_{qs} + [n_1]_{qs} + [n_2 + 1]_{qs^{-1}} + [n_2]_{qs^{-1}} \\ &+ [n_3 + 1]_{qs} + [n_3]_{qs} + [n_4 + 1]_{qs^{-1}} + [n_4]_{qs^{-1}}\} = e^2\end{aligned} \tag{37}$$

From Eqs. (34) and (37), we have the energy spectrum of the q, s-deformed hydrogen atom,

$$E_{qs} = \frac{-\mu e^4}{\begin{aligned}2\hbar^2\{&(1/4)([n_1 + 1]_{qs} + [n_1]_{qs} + [n_2 + 1]_{qs^{-1}} \\ &+ [n_2]_{qs^{-1}} + [n_3 + 1]_{qs} + [n_3]_{qs} + [n_4 + 1]_{qs^{-1}} + [n_4]_{qs^{-1}})\}^2\end{aligned}} \tag{38}$$

On the other hand, the constraint condition is

$$s^{n_1-n_2}\left\{s^2[n_1+1]_{qs}[n_2]_{qs^{-1}}+\left[\frac{n_1-n_2}{2}\right]_{qs}\left[\frac{n_1-n_2}{2}+1\right]_{qs}\right\}$$

$$=s^{n_3-n_4}\left\{s^2[n_3+1]_{qs}[n_4]_{qs^{-1}}+\left[\frac{n_3-n_4}{2}\right]_{qs}\left[\frac{n_3-n_4}{2}+1\right]_{qs}\right\} \quad (39)$$

In particular, Eq. (38) reduces to the general case of the hydrogen atom as $q \to 1$ and $s \to 1$.

REFERENCES

Gerry, C. C. (1986). *Phys. Rev. A* **33**, 6.
Jing, S. C., and Cuypers, F. (1993). *Commun. Theor. Phys.* **19**, 495.
Kibler, M., and Négadi, T. (1991). *J. Phys. A* **21**, 5283.
Kustaanheimo, P., and Stiefel, E. (1965). *J. Reine Angew. Math.* **1965**, 218.